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Some General Characterizations of the Bivariate Gumbel Distribution and the Bivariate Lomax Distribution Based on Truncated Expectations

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Recently attempts have been made to characterize probability distributions via truncated expectations in both univariate and multivariate cases. In this paper we will use a well known theorem of Lau and Rao (1982) to obtain some characterization results, based on the truncated expectations of a function h, for the bivariate

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1. INTRODUCTION

In recent years, a result of Lau and Rao (1982) on the integrated Cauchy functional equation, which is known as the Lau–Rao theorem, has proved to be a powerful tool in characterization theory. This theorem was motivated by a result of Shanbhag (1977), which is known as Shanbhag's lemma, and subsumes some partial results given earlier by Shimizu (1978) and Ramachandaran (1982). Many characterization results, especially those for the exponential and geometric distributions, based on the truncated expectations, order statistics and record values follow as corollaries of the Lau–Rao theorem. For the relevant details one can see either the monograph of Rao and Shanbhag (1994) or the paper of Rao and Shanbhag (1986). An elegant probabilistic proof based on exchangeability for the Lau–Rao theorem appears in Alzaid *et al.* (1987).

In reliability theory, in studies of the lifetime of a component or a system, a flexible model which has been widely used in the literature is that of a generalized Pareto distribution (GPD). This model has been introduced by Hall and Wellner (1981), and includes the exponential distribution, the Lomax distribution and the power distribution as special cases. The latter authors showed that two characterizations of this model are linearity of the mean residual life function and constancy of the coefficient of

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variation of the residual life distribution. Mukherjee and Roy (1986) showed that the constancy of the product of the mean residual life function and the hazard rate function implies that the distribution is GPD. Oakes and Dasu (1990) gave a different characterization result for GPD, of the type of lack of memory property of the exponential distribution. Korwar (1992) obtained a characterization result of this family based on the skewness and the kurtosis of the residual life distribution. Asadi *et al.* (1997) derived some general characterizations of the GPD family based on the truncated expectations, order statistics and record values. In particular they proved that if a function h, meeting some conditions, satisfies $E[h((X-x)/(cx+1)) \mid X \geqslant x] = E[h(X)]$ then the underling distribution is GPD. This result includes a result of Hall and Wellner (1981) as a special case.

In the bivariate set-up, a family of distributions appearing below have properties analogous to GPD concerning reliability measures.

- The bivariate exponential distribution due to Gumbel (1960) with survival function

$$\bar{F}(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_1 x_2}, \qquad x_1, x_2 \geqslant 0, \tag{1}$$

where $\lambda_1, \lambda_2 > 0$, and $0 \leqslant \lambda_3 \leqslant \lambda_1 \lambda_2$.

- The bivariate Lomax distribution with survival function

$$\bar{F}(x_1, x_2) = (1 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1 x_2)^{-q}, \qquad x_1, x_2 \geqslant 0, \tag{2}$$

where $q > 0$, $\lambda_1, \lambda_2 > 0$, and $0 \leqslant \lambda_3 \leqslant \lambda_1 \lambda_2 (1 + q)$,

- The bivariate power distribution with survival function

$$\bar{F}(x_1, x_2) = (1 - \lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_1 x_2)^{q}, \tag{3}$$

where $q > 0$, $0 \leqslant x_1 \leqslant \lambda_1^{-1}$, $0 \leqslant x_2 \leqslant (1 - \lambda_1 x_1)/(\lambda_1 + \lambda_3 x_1)$, and $\lambda_1, \lambda_2 > 0$, $-\lambda_1 \lambda_2 \leqslant \lambda_3 \leqslant \lambda_1 \lambda_2 (q - 1)$.

Roy (1989, 1990) obtained an extended version of the result of Mukherjee and Roy (1986) characterizing these three bivariate distributions. Asadi (1997) extended stronger versions of the results of Roy (1989, 1990) to the multivariate case. Roy and Gupta (1996) arrived at an analogue of the result of Hall and Wellner (1981) on constancy of the coefficient of variation of the residual life for these bivariate distributions.

In the present paper, using a corollary of the Lau–Rao theorem, we shall extend a result of Asadi *et al.* (1997) on GPD family, to the bivariate case and arrive at some general characterization results based on truncated expectation of a function h, meeting some conditions. Also we obtain some characterization results for the above bivariate distributions based on the coefficient of variation of the residual life; these results are stronger than those obtained earlier by Roy and Gupta (1996).

2. SOME BASIC TOOLS

In the following, we shall state the Lau–Rao theorem and prove a corollary of that; we refer to these in our derivations of the main results.

THEOREM 1 (Lau and Rao, 1982). *Let H: $R_+ \to R_+$ be a Borel measurable function that is locally integrable (w.r.t. the Lebesgue measure) and is not zero a.e. (L) (i.e., almost everywhere w.r.t. the Lebesgue measure). Let H satisfies in integral equation*

$$H(x) = \int_{R_+} H(x+y)\,\mu(dy), \qquad x \in R_+,$$

where μ is a σ-finite measure on R_+ satisfying $\mu\{0\} < 1$. Then, either μ is arithmetic with some span λ and

$$H(x+n\lambda) = H(x)\,b^n \qquad n = 0, 1, \ldots \text{ for almost all } [L]\ x \in R_+$$

with b such that

$$\sum_{n=0}^{\infty} b^n \mu(n\lambda) = 1,$$

or μ is non-arithmetic and

$$H(x) \ \propto\ \exp(-\lambda x) \qquad \text{for almost all } [L]\ x \in R_+,$$

with λ such that

$$\int_{R_+} \exp(-\lambda x)\,\mu(dx) = 1.$$

There are many proofs of this theorem in the literature. For a recent proof of the theorem based on exchangeability one can see Alzaid *et al.* (1987), or Rao and Shanbhag (1994).

In the following, using the Lau–Rao theorem, we prove a theorem which is useful in obtaining of our characterizations results.

THEOREM 2. *Let H and μ defined as above theorem, μ be non-atomic and H satisfies in following equation*

$$H(x) = \int_{R_+} H[x + (\alpha x + \beta)\,y]\,\mu(dy), \qquad x \in C,$$

where α is real valued and $\beta > 0$.

(i) *If* $\alpha > 0$ *and* $C = R_+$, *then*

$$H(x) \;\propto\; \left(x + \frac{\beta}{\alpha}\right)^{-\lambda}, \qquad x \in R_+,$$

with λ *such that*

$$\int_{R_+} (1 + \alpha y)^{-\lambda}\, \mu(dy) = 1.$$

(ii) *if* $\alpha < 0$, $c = (0, -\beta/\alpha)$ *and for each support point* s *of* μ, $0 < \alpha s + 1 \leqslant 1$, *then*

$$H(x) \;\propto\; \left(-x - \frac{\beta}{\alpha}\right)^{\lambda}, \qquad x \in \left(0, -\frac{\beta}{\alpha}\right),$$

with λ *such that*

$$\int_0^{-\beta/\alpha} (1 + \alpha y)^{\lambda}\, \mu(dy) = 1.$$

Proof. Note that if $\alpha = 0$ and $\beta = 1$ then the theorem reduces to the Lau–Rao theorem.

(i) Let $\alpha > 0$. Take $H(x) = H^*(x + \beta/\alpha)$, then

$$\begin{aligned} H^*\left(x + \frac{\beta}{\alpha}\right) &= \int_{R_+} H^*\left[x + (\alpha x + \beta)\, y + \frac{\beta}{\alpha}\right] \mu(dy) \\ &= \int_{R_+} H^*\left[\left(x + \frac{\beta}{\alpha}\right)(\alpha y + 1)\right] \mu(dy) \\ &= \int_{R_+} H^*[e^{\ln(x + (\beta/\alpha)) + \ln(\alpha y + 1)}]\, \mu(dy). \end{aligned}$$

Since we assume that μ is non-atomic, we have the measure induced by $\ln(\alpha y + 1)$ to be non-atomic. Using then the Lau–Rao theorem, we see that for $x \in R_+$

$$H(x) \;\propto\; e^{-\lambda \ln(x + (\beta/\alpha))} = \left(x + \frac{\beta}{\alpha}\right)^{-\lambda}.$$

(ii) Let $\alpha < 0$, with the support point condition under (ii) met. On taking $H(x) = H^*(-x - (\beta/\alpha))$ we have

$$H^*\left(-x - \frac{\beta}{\alpha}\right) = \int_{R_+} H^*\left[-\left(x + \frac{\beta}{\alpha}\right)(\alpha y + 1)\right] \mu(dy).$$

This implies in view of the Lau–Rao theorem for $x \in (0, -\beta/\alpha)$

$$\begin{aligned} H(x) &\propto e^{\lambda \ln(-x-(\beta/\alpha))} \\ &= \left(-x - \frac{\beta}{\alpha}\right)^{\lambda}. \end{aligned}$$

This completes the theorem.

3. CHARACTERIZATION BASED ON TRUNCATED EXPECTATIONS

In this section we shall prove two general characterization theorems in bivariate case, which are extensions of the results obtained by Asadi *et al.* (1997) in the univariate case.

THEOREM 3. *Let $X = (X_1, X_2)$ be a non-negative bivariate random vector with continuous distribution function F. Let h be a monotone right continuous function on R_+ such that $E[|h(X_i)|] < \infty$ and $E[h(X_i)] \neq h(0)$, $i = 1, 2$. Consider the following equations*

$$E\left[h\left(\frac{X_1 - x_1}{\theta(x_2)}\right) \,\middle|\, X_1 \geqslant x_1, X_2 \geqslant x_2\right] = E[h(X_1)], \tag{4}$$

$$E[h(X_2 - x_2) \mid X_2 \geqslant x_2] = E[h(X_2)], \tag{5}$$

where $\theta(x_2) = a/(a + bx_2)$ $0 < a$, $0 \leqslant b$. Let $h_i^*(x)$ be a non-arithmetic function, where

$$h_i^*(x) = \frac{h(x) - h(0)}{E[h(X_i)] - h(0)} \qquad x \in R_+ \quad i = 1, 2.$$

Then X has a bivariate Gumbel distribution of the form (1) *if and only if conditions* (4) *and* (5) *hold.*

Proof. Without loss of generality we assume that h to be increasing. Now let the equation (4) hold, i.e.,

$$\int_{[x_1, \infty)} h\left(\frac{y_1 - x_1}{\theta(x_2)}\right) d\bar{F}_1(y_1 \mid x_2) = E[h(X_1)]\, \bar{F}_1(x_1 \mid x_2), \qquad x \in R_+.$$

This equation implies, via Fubini's theorem, that for each $x_2 \geqslant 0$,

$$\int_{[0, \infty)} \bar{F}_1[x_1 + \theta(x_2)\, y \mid x_2]\, dh_1^*(y) = \bar{F}_1(x_1 \mid x_2), \qquad x \in R_+ \tag{6}$$

where h_1^* is as in the statement of the theorem and $\bar{F}_1(\cdot \mid \cdot)$ denotes the conditional survival function of X_1, given $X_2 \geqslant x_2$. Now appealing to the Lau–Rao theorem, we get from (6) that there exists a function $\beta(x_2) < 0$ such that

$$\bar{F}_1(x_1 \mid x_2) = e^{\beta(x_2)\, x_1}, \qquad x_1, x_2 \geqslant 0$$

with $\beta(x_2)$ satisfying

$$\int_{[0, \infty)} e^{\{\beta(x_2)\, \theta(x_2)\}\, y}\, dh_1^*(y) = 1. \tag{7}$$

The uniqueness of Laplace transform implies that in (7), $\beta(x_2)\, \theta(x_2)$ must be a negative constant independent of x_2. Consequently, in view of the definition of $\theta(x_2)$ we get that $\beta(x_2)$ must be of the form

$$\beta(x_2) = -(\lambda_1 + \lambda_3 x_2), \qquad \lambda_1 > 0, \quad \lambda_3 \geqslant 0, \quad x_2 \geqslant 0,$$

(with obviously $\lambda_3/\lambda_1 = b/a$). This implies

$$\bar{F}(x_1 \mid x_2) = e^{-(\lambda_1 + \lambda_3 x_2)\, x_1}. \tag{8}$$

Now let us have the equation (5). Using Theorem 5.2.6 in Rao and Shanbhag (1994, p. 108) we get that X_2 is an exponential random variable with parameter λ_2 (say). Hence in view of this and equation (8) we obtain

$$\bar{F}(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2 - \lambda_3 x_1 x_2} \qquad x_1 \geqslant 0, \quad x_2 \geqslant 0.$$

Since $\bar{F}(x_1, x_2)$ is a bivariate survival function it follows that $0 < \lambda_1$, $\lambda_2 < \infty$ and $0 \leqslant \lambda_3 \leqslant \lambda_1 \lambda_2$. Hence $X = (X_1, X_2)$ is a vector distributed according to a Gumbel bivariate distribution. The converse of the theorem is easy to prove. Hence we have the theorem.

Remark 1. If we assume that X has a bivariate Gumbel distribution of the form (1), then $\theta(x_2)$ in Theorem 3 equals $m_1(x_1, x_2)/m(0,0)$, where $m_1(x_1, x_2) = E(X_1 - x_1 \mid X_1 \geqslant x_1, X_2 \geqslant x_2)$.

Theorem 1 gives the following corollary:

COROLLARY 1. *The random vector $X = (X_1, X_2)$ has a bivariate Gumbel distribution if and only if*

$$E(X_1 - x_1 \mid X_1 \geqslant x_1, X_2 \geqslant x_2) = E(X_1)\, \theta(x_2), \qquad x_1, x_2 \geqslant 0 \tag{9}$$

$$E(X_2 - x_2 \mid X_2 \geqslant x_2) = E(X_2), \qquad x_2 \geqslant 0, \tag{10}$$

where $\theta(x) = \lambda_1/(\lambda_1 + \lambda_3 x_2)$.

Proof. The result follows from Theorem 3, on taking $h(x) = x$.

This result shows that when the mean residual life function of X_i, given $X_{3-i} \geqslant x_{3-i}$ and the mean residual life function of X_{3-i}, $i = 1, 2$, are both constant, then the vector $X = (X_1, X_2)$ has the bivariate Gumbel distribution.

In the following theorem we shall prove a similar result characterizing a bivariate Lomax distribution.

THEOREM 4. *Let conditions of the Theorem* 3 *hold and consider the following equations*

$$E\left[h\left(\frac{X_1 - x_1}{\theta_1(x_1, x_2)} \right) \middle| X_1 \geqslant x_1, X_2 \geqslant x_2 \right] = E[h(X_1)], \qquad x_1, x_2 \geqslant 0 \tag{11}$$

$$E\left[h\left(\frac{X_2 - x_2}{\theta_2(c_i, x_2)} \right) \middle| X_1 \geqslant c_i, X_2 \geqslant x_2 \right] = E[h(X_2)], \qquad x_2 \geqslant 0, \tag{12}$$

where c_i, $i = 1, 2$, are non-negative constants,

$$\theta_1(x_1, x_2) = \left(x_1 + \frac{1 + \lambda_2 x_2}{\lambda_1 + \lambda_3 x_2} \right) \lambda_1$$

and

$$\theta_2(x_1, x_2) = \left(x_2 + \frac{1 + \lambda_1 x_1}{\lambda_2 + \lambda_3 x_1} \right) \lambda_2.$$

Then the random vector X has the bivariate Lomax distribution of the form (2) *if and only if equations* (11) *and* (12) *hold.*

Proof. First we prove the "if" part of the theorem. Let the equations (11) and (12) be valid. In view of Fubini's theorem, we can show that for each $x_2 \geqslant 0$

$$\int_{[0,\,\infty)} \bar{F}_1[\, y(\lambda_1 x_1 + \lambda_1 \phi_1(x_2)) + x_1 \mid x_2]\, dh_1^*(y) = \bar{F}_1(x_1 \mid x_2), \qquad x_1 \geqslant 0 \tag{13}$$

$$\int_{[0,\,\infty)} \bar{F}_2[\, y(\lambda_2 x_2 + \lambda_2 \phi_2(c_i)) + x_2 \mid c_i)\, dh_2^*(y) = \bar{F}_2(x_2 \mid c_i) \qquad i = 1, 2, \tag{14}$$

where h_i^*, $i=1,2$ are as defined in the Theorem 3, $\phi_1(x_2) = (1+\lambda_2 x_2)/(\lambda_1 + \lambda_3 x_2)$ and $\phi_2(c_i) = (1+\lambda_1 c_i)/(\lambda_2 + \lambda_3 c_i)$. Now, in view of Theorem 2, there exist functions α_i and β_i, $i=1,2$, such that

$$\bar{F}_1(x_1 \mid x_2) = \alpha_1(x_2)(\lambda_1 x_1 + \lambda_1 \phi_1(x_2))^{\beta_1(x_2)}$$
$$\bar{F}_2(x_2 \mid c_i) = \alpha_2(c_i)(\lambda_2 x_2 + \lambda_2 \phi_2(c_i))^{\beta_2(c_i)},$$

where $\alpha_i > 0$ and β_i, $i=1,2$ are such that

$$\int_{[0,\,\infty)} (1+\lambda_i y)^{\beta_i(\cdot)}\, dh_i^*(y) = 1, \tag{15}$$

but $\bar{F}_i(0 \mid \cdot) = 1$ implies, $\alpha_i(\cdot) = (1/\lambda_i \phi_i(\cdot))^{\beta_i(\cdot)}$. Using the arguments that were used in the last theorem, it follows that the equation (15) implies that $\beta_1(\cdot)$ and $\beta_2(\cdot)$ must be negative constants q_1 and q_2 (say), independent of x_2 and c_i, respectively. Hence, we have

$$\bar{F}(x_1, x_2) = \bar{F}_2(x_2)\left(\frac{x_1}{\phi_1(x_2)} + 1\right)^{q_1} = \bar{F}_2(x_2)\left(\frac{1+\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1 x_2}{1+\lambda_2 x_2}\right)^{q_1} \tag{16}$$

and

$$\bar{F}(c_i, x_2) = \bar{F}_1(c_i)\left(\frac{x_2}{\phi_2(c_i)} + 1\right)^{q_2} = \bar{F}_1(c_i)\left(\frac{1+\lambda_1 c_i + \lambda_2 x_2 + \lambda_3 c_i x_2}{1+\lambda_1 c_i}\right)^{q_2}. \tag{17}$$

Using the equations (16) and (17) respectively, we obtain

$$\frac{\bar{F}(c_1, x_2)}{\bar{F}(c_2, x_2)} = \left(\frac{1+\lambda_1 c_1 + \lambda_2 x_2 + \lambda_3 c_1 x_2}{1+\lambda_1 c_2 + \lambda_2 x_2 + \lambda_3 c_2 x_2}\right)^{q_1} \tag{18}$$

$$\frac{\bar{F}(c_1, x_2)}{\bar{F}(c_2, x_2)} = k\left(\frac{1+\lambda_1 c_1 + \lambda_2 x_2 + \lambda_3 c_1 x_2}{1+\lambda_1 c_2 + \lambda_2 x_2 + \lambda_3 c_2 x_2}\right)^{q_2}, \tag{19}$$

where k is a function of c_1 and c_2. The equality of (18) and (19) implies that $q_1 = q_2 = q$ (say). On the other hand, we see from (16) (with $x_1 = c_i$) and (17) that

$$\bar{F}_2(x_2) = (1+\lambda_2 x_2)^q \qquad x_2 \geqslant 0.$$

Hence, from (16) we obtain that

$$\bar{F}(x_1, x_2) = (1+\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_1 x_2)^q, \qquad x_1, x_2 \geqslant 0. \tag{20}$$

Since $\bar{F}$ is a bivariate survival function, we must have $0 < \lambda_1,\ \lambda_2 < \infty$ and $0 \leqslant \lambda_3 \leqslant \lambda_1 \lambda_2 (q+1)$. This completes the "if" part of the theorem. The "only if" part of the theorem is easy to prove and we have the theorem.

Theorem 4 gives the following corollary

COROLLARY 2. *The random vector $X = (X_1, X_2)$ has a bivariate Lomax distribution of the form* (2) *if and only if*

$$E(X_1 - x_1 \mid X_1 \geqslant x_1, X_2 \geqslant x_2) = \left(x_1 + \frac{1+\lambda_2 x_2}{\lambda_1 + \lambda_3 x_2}\right)\lambda_1 E(X_1)$$

and

$$E(X_2 - x_2 \mid X_1 \geqslant c_i, X_2 \geqslant x_2) = \left(x_2 + \frac{1+\lambda_1 c_i}{\lambda_2 + \lambda_3 c_i}\right)\lambda_2 E(X_2).$$

Proof. The result follows from Theorem 4 on taking $h(x) = x$.

This shows that a random vector $X = (X_1, X_2)$ has a bivariate Lomax distribution if and only if its conditional mean residual life functions are locally linear.

In the following theorem, we state an analogous theorem characterizing the bivariate power distribution with survival function of the form (3). The proof of the result is similar, on using Theorem 2, to that of Theorem 4 and hence is omitted.

THEOREM 5. *Let conditions of Theorem* 4 *hold and consider the following equations*

$$E\left[h\left(\frac{X_1 - x_1}{\theta_1(x_1, x_2)}\right) \middle| X_1 \geqslant x_1, X_2 \geqslant x_2 \right] = E[h(X_1)] \quad x_1, x_2 \geqslant 0, \tag{21}$$

$$E\left[h\left(\frac{X_2 - x_2}{\theta_2(c_i, x_2)}\right) \middle| X_1 \geqslant c_i, X_2 \geqslant x_2 \right] = E[h(X_2)], \quad x_2 \geqslant 0 \tag{22}$$

where c_i, $i=1, 2$, *are non-negative constants,* h_i^*, $i=1, 2$, *are as defined in Theorem* 4,

$$\theta_1(x_1, x_2) = \left(\frac{1 - \lambda_2 x_2}{\lambda_1 + \lambda_3 x_2} - x_1\right)\lambda_1$$

and

$$\theta_2(x_1, x_2) = \left(\frac{1 - \lambda_1 x_1}{\lambda_2 + \lambda_3 x_1} - x_2\right)\lambda_2.$$

Then the random vector X *has the bivariate power distribution with survival function of the form* (3) *if and only if equations* (21) *and* (22) *hold.*

COROLLARY 3. *The random vector* $X = (X_1, X_2)$ *has the bivariate power distribution with survival function* (3) *if and only if*

$$E(X_1 - x_1 \mid X_1 \geqslant x_1, X_2 \geqslant x_2) = \left(\frac{1 - \lambda_2 x_2}{\lambda_1 + \lambda_3 x_2} - x\right)\lambda_1 E(X_1)$$

and

$$E(X_2 - x_2 \mid X_1 \geqslant c_i, X_2 \geqslant x_2) = \left(\frac{1 - \lambda_1 c_i}{\lambda_2 + \lambda_3 c_i} - c_2\right)\lambda_2 E(X_2).$$

Proof. The result follows from Theorem 5 on taking $h(x) = x$.

4. CHARACTERIZATION BASED ON THE COEFFICIENT OF VARIATION OF THE RESIDUAL LIFE

Characterizations of the life distributions based on the coefficient of variation of the residual life (CVRL) has also been considered in the literature. Let X be a non-negative random variable with mean residual life function

(MRL) $m(x) = E[X - x \mid X \geqslant x]$ and variance residual life function (VRL) $v(x) = E[(X-x)^2 \mid X \geqslant x] - m^2(x)$. The CVRL of X is defined for any $x \geqslant 0$ as

$$C(x) = \frac{v^{1/2}(x)}{m(x)}. \tag{23}$$

Hall and Wellner (1981) showed that a random variable X has a constant CVRL if and only the underling distribution is GPD (see also Mukherjee and Roy, 1986). Roy and Gupta (1996) extended this result to the bivariate case, characterizing the distributions with survival functions (1), (2) and (3). In this section we will give some characterization results based on CVRL in a bivariate set up; these results are stronger than those gave by Roy and Gupta (1996).

Let $X = (X_1, X_2)$ be a non-negative random vector with bivariate MRL $m = (m_1, m_2)$, bivariate VRL $v = (v_1, v_2)$ and bivariate CVRL $C = (C_1, C_2)$, where

$$m_i(x_1, x_2) = E(X_i - x_i \mid X_1 \geqslant x_1, X_2 \geqslant x_2) \qquad x_1, x_2 \geqslant 0, \quad i = 1, 2, \tag{24}$$

$$v_i(x_1, x_2) = \operatorname{Var}(X_i - x_i \mid X_1 \geqslant x_1, x_2 \geqslant x_2) \qquad x_1, x_2 \geqslant 0, \quad i = 1, 2, \tag{25}$$

and

$$C_i(x_1, x_2) = \frac{v_i^{1/2}(x_1, x_2)}{m_i(x_1, x_2)} \qquad x_1, x_2 \geqslant 0, i = 1, 2. \tag{26}$$

Note that $m_i(x_1, x_2)$, $v_i(x_1, x_2)$ and $C_i(x_1, x_2)$ denote respectively the MRL, VRL and CVRL of the conditional random variable X_i given $X_{3-i} \geqslant x_{3-i}$, $i = 1, 2$.

Theorem 6. *Let $X = (X_1, X_2)$ be a non-negative continuous random vector with CVRL $C = (C_1, C_2)$. Then*

$$C_1(x_1, x_2) = 1 \tag{27}$$

and

$$C_2(c_i, x_2) = 1, \qquad i = 1, 2, \tag{28}$$

where c_i are non-negative constants, if and only if $X = (X_1, X_2)$ has the bivariate Gumbel distribution.

Proof. Using the result of Hall and Wellner (1981), we obtain from (27) and (28) that the random variable X_1, given $X_2 \geqslant x_2$ and the random variable X_2, given $X_1 \geqslant c_i$ are distributed exponentially with survival functions

$$\bar{F}_1(x_1 \mid x_2) = e^{-\theta_1(x_2)\, x_1}, \qquad x_1, x_2 \geqslant 0$$

and

$$\bar{F}_2(x_2 \mid c_i) = e^{-\theta_2(c_i)\, x_2}, \qquad x_2 \geqslant 0,$$

respectively, where $\theta_1(x_2)$ and $\theta_1(c_i)$ may depend on x_2 and c_i, respectively. Now the result follows easily from Theorem 1, Asadi (1997).

In the following theorem, we give a similar result characterizing the bivariate Lomax distribution.

THEOREM 7. *Let $X = (X_1, X_2)$ be a non-negative continuous random vector with CVRL $C = (C_1, C_2)$. Further let c_i, $i = 1, 2$ and k be non-negative constants, such that $k > 1$. Then*

$$C_1(x_1, x_2) = k \tag{29}$$

and

$$C_2(c_i, x_2) = k \tag{30}$$

if and only if $X = (X_1, X_2)$ has a bivariate Lomax distribution with survival function of the form (2).

Proof. Let conditions (29) and (30) be valid. From the result of Hall and Wellner (1981), we have this to be equivalent to the assertions that

$$m_1(x_1, x_2) = \alpha x_1 + R_1(x_2)$$

and

$$m_2(c_i, x_2) = \alpha x_2 + R_2(c_i),$$

where α is a positive constant depending on k (see also Lemma 2.3 of Roy and Gupta, 1996). Hence the result follows easily from Theorem 1, Asadi (1997).

Remark 2. In Theorem 7 if we assume that $k < 1$, then with the same arguments we get that $X = (X_1, X_2)$ has the bivariate power distribution with survival function of the form (3).

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